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Mixed H_2/H_∞ Control Design for Time Delay Systems: An LMI Approach

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Abstract

This study introduces a mixed H_2/H_∞ control design for uncertain time delay systems with guaranteed control performance. Based on Lyapunov criterion and Krasovskii stability theorem, a sufficient condition is derived under the uncertain parameters and external disturbances. By Schur complement, the sufficient condition can be easily transformed into the problems of linear matrix inequality (LMI). The mixed H_2/H_∞ performance problems in this study is characterized in terms of two eigenvalue problems (EVPs). The EVPs can be solved very efficiently by the convex optimization techniques. Simulation results indicate that the desired mixed H_2/H_∞ performance for uncertain time delay systems can be achieved via the proposed method.

Keyword: delay systems, linear matrix inequality, mixed H_2/H_∞ control, Lyapunov-Krasovskii functional

針對時間延遲系統的混合型H_2/H_∞控制設計:線性矩陣不等式處理

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摘要

本篇論文針對不確定性時間延遲系統介紹混合型H_2/H_∞控制設計以保證控制性能。根據Lyapunov 準則與 Krasovskii 穩定定理，在不確定參數與外來干擾下，一個充分條件被導出。藉由 Schur 補數，充分條件很容易轉換成線性矩陣不等式（LMI）的問題。本論文中混合型H_2/H_∞性能問題可以特性化爲兩個特徵值型式的問題。而特徵值可以充分藉由凸形最佳化技巧求解。模擬結果指出不確定時間延遲系統所要求的混合型H_2/H_∞性能藉由所陳述的方法能夠被達成。

關鍵字：延遲系統, 線性矩陣不等式, 混合型H_2/H_∞控制, Lyapunov-Krasovskii 函數

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I. Introduction

Time delay is encountered in many fields of engineering and science, such as communication networks, manufacturing systems, chemical processes, rolling mill systems, hydraulic systems, turbojet engine, microwave oscillator, nuclear reactor, long transmission lines, life science and economics. The systems with time delayed state, perturbation and external disturbance are problems of theoretical and practical interest, since the existence of delay, perturbation and disturbance are often sources of instability.

In the past decade, many researchers have paid a great deal of attention to various control methods in time delay systems and criteria for the stability of time delay systems. Stability criteria for time delay systems can be classified into two categories: 1) there is no information about the size of delay, i.e., delay-independent criteria [1, 2, 3, 8]; 2) there is some information about the size of delay, i.e., delay-dependent criteria [4, 5, 6, 9, 10].

In [7]-[10], the control design for time delay systems via Lyapunov-Krasovskii functional approach, and their conditions are expressed in terms of finite-dimensional Riccati equations [12], finite-dimensional Lyapunov equations [11], or matrix inequalities [8, 9, 10, 13]. The derived conditions guarantee the delay-independent [8, 13] or delay-dependent [9, 10, 12] asymptotic stability of the closed-loop system.

Recently, the robust H_∞ control problem for time delay systems has been proposed. The existing results for robust control of time delay systems deal with either one of two types of stabilization: delay-dependent stabilization and delay-independent stabilization. In [8], the H_∞ memoryless control and α-stability constrained for time delay systems via Lyapunov-Krasovskii functional and LMI approach has been proposed. In [10], the delay dependent robust H_∞ control for uncertain time delay systems are developed.

The primary contribution of this paper is that the mixed H_2/H_∞ control design and the robust performance for the time delay systems is guaranteed. A robust stabilization technique is also proposed to override the bounded nonlinear perturbation. Furthermore, an LMI optimization technique is developed to solve the mixed H_2/H_∞ control problem.

Simulation examples are provided to illustrate the design procedure and the performance of the proposed method. The simulation results show that the robustness performance can be achieved by the proposed method.

The paper is organized as follows. The problem formulation is presented in Section II. In Section III, a mixed H_2/H_∞ control design is introduced. In Section IV, simulation examples are provided to demonstrate the design effectiveness and to confirm the desired performance. Finally, concluding remarks are made in Section V.

II. Problem Formulation

Consider the following uncertain linear system with delayed state as follows:

$$\dot{x}(t) = (A+\Delta A)x(t) + (A_d+\Delta A_d)x(t-\tau) + (B_u+\Delta B_u)u(t) + B_w w(t) \tag{1}$$

where $x(t)=\Psi(t)$, $t\in[-\tau,0]$, and A, A_d, B_u and B_w are constant matrices with appropriate dimensions, $x(t)\in R^n$ is the state vector, $u(t)\in R^m$ is the input vector, $w(t)\in R^w$ is the external disturbance vector and $\Psi(t)$ is the initial condition of the state, ΔA, ΔA_d and ΔB_u are bounded nonlinear perturbation matrices.

Suppose the following robust H_∞ controller is employed to deal with the above time delay system as follows

$$u(t) = -Kx(t) \tag{2}$$

Substituting (2) into (1) yields the closed-loop time delay system as follows:

$$\begin{aligned}\dot{x}(t) &= (A+\Delta A)x(t)+(A_d+\Delta A_d)x(t-\tau)+(B_u+\Delta B_u)u(t)+B_w w(t)\\ &= (A-B_uK)\,x(t)+A_d x(t-\tau)+\Delta A x(t)+\Delta A_d x(t-\tau)\\ &\quad -\Delta B_u K x(t)+B_w w(t)\end{aligned} \tag{3}$$

Suppose that there exist bounding matrices A_p, A_{dp} and B_p such that

$$\|\Delta A x(t)\| \leq \|A_p x(t)\|\,, \tag{4}$$

$$\|\Delta A_d x(t-\tau)\| \leq \|A_{dp} x(t-\tau)\|\,, \tag{5}$$

and

$$\|\Delta B_u K x(t)\| \leq \|B_p K x(t)\|\,, \tag{6}$$

for all trajectory $x(t)$.

According to assumption above, we get

$$\begin{aligned}(\Delta A x(t))^T(\Delta A x(t)) &\leq (A_p x(t))^T(A_p x(t))\,,\\ (\Delta A_d x(t-\tau))^T(\Delta A_d x(t-\tau)) &\leq (A_{dp} x(t-\tau))^T(A_{dp} x(t-\tau))\,,\end{aligned}$$

and

$$(\Delta B_u K x(t))^T(\Delta B_u K x(t)) \leq (B_p K x(t))^T(B_p K x(t))\,.$$

Assumption : *We assume the initial condition as follows*

$$\Psi(t)=x(0),\ \forall t\in[-\tau,0]\,. \tag{7}$$

In this study, we assume that $w(t)$ is uncertain but bounded. However, the effect of $w(t)$ will deteriorate the control performance of the control system and even lead to instability of the time delay control system. Therefore, how to eliminate the effect of $w(t)$ to guarantee control performance is also an important design purpose of robust control systems. Since H_∞ control is the most important control design to efficiently eliminate the effect of $w(t)$ on the control system, it will be employed to deal with the robust performance control in (1). Let us consider the following H_2 and H_∞ control performances.

■ H_∞ Performance: An H_∞ performance is considered as follows [14, 15]:

$$\frac{\int_0^{t_f} x^T(t)Qx(t)dt}{\int_0^{t_f} w^T(t)w(t)dt} < \rho^2 \tag{8}$$

or

$$\int_0^{t_f} x^T(t)Qx(t)dt < \rho^2 \int_0^{t_f} w^T(t)w(t)dt \tag{9}$$

where t_f denotes the terminal time of control, ρ is a prescribed value which denotes the worst case effect of $w(t)$ on $x(t)$, and Q is a positive-definite weighting matrix. The physical meaning of (9) is that the effect of $w(t)$ on $x(t)$ must be attenuated below a desired level ρ from the viewpoint of energy, no matter what $w(t)$ is, i.e., the L_2 gain from $w(t)$ to $x(t)$ must be equal to or less than a prescribed value ρ^2. In general, ρ is chosen as a positive small value less than 1 for attenuation of $w(t)$.

The inequality in (9) can be seen as bounded-disturbance and bounded-state but with a prescribed gain ρ. If the initial condition is also considered, the inequality (9) can be modified as

$$\int_0^{t_f} x^T(t)Qx(t)dt \leq x^T(0)(P+\tau S)x(0)+\rho^2\int_0^{t_f} w^T(t)w(t)dt \tag{10}$$

where P is some symmetric positive-definite weighting matrix, and $S = S^T > 0$ is a weighting matrix.

From the analysis above, the design purpose of the proposed control system is to specify a linear control (2) such that both the stability of linear control system and the H_∞ control performance in (10) with a prescribed attenuation level ρ are guaranteed.

The robustness optimization is to achieve a minimum ρ^2 in (10) to obtain maximum elimination of the effect of $w(t)$. For time-delay system (1), this design problem is how to specify a stabilizable linear control in (2) to minimize ρ^2 subject to the constraint (10).

■ H_2 Performance: An LQ performance related to the tracking error and control action is considered as followings:

$$J_2 = \int_0^{t_f} \left[e^T(t)Q_2e(t) + u^T(t)R_2u(t)\right] dt \tag{11}$$

where $Q_2 > 0$ and $R_2 > 0$. A straightforward objective is to minimize this performance. However, this objective is not easily tractable, since uncertainties involve in tracking error dynamics. Therefore, in this study, a suboptimal approach is taken by minimizing the upper bound of performance index.

III. Mixed H_2/H_∞ control design

The design purpose in this study is to specify the control in (2) to achieve the mixed H_2/H_∞ control performance in (11) and (10) simultaneously. The design procedure is discussed step by step as the following. First, robust H_∞ control design and the H_2 control design procedure is discussed. Then, the problem of mixed H_2/H_∞ control design is parameterized in terms of two EVP's.

A. Robust H_∞ Control Design

From the analysis above, the first step in the design process of the time delay systems is to specify a controller such that the system is robustly stabilized and the effect of external disturbance $w(t)$ is efficiently attenuated, thus achieving H_∞ control performance in (10) with a prescribed attenuation level ρ.

Let us consider the following Lyapunov-Krasovskii functional candidate:

$$V(t, x(t)) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(\theta)Sx(\theta)d\theta \tag{12}$$

where the weighting matrices P and S are the same as that in (10).

From (12), we obtain

$$\begin{aligned}
& \int_0^{t_f} x^T(t)Qx(t)dt \\
= \; & V(0, x(0)) - V(t_f, x(t_f)) + \int_0^{t_f} [x^T(t)Qx(t) + \dot{V}(t, x(t))]dt \\
= \; & V(0, x(0)) - V(t_f, x(t_f)) + \int_0^{t_f} [x^T(t)Qx(t) \\
& + \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + x^T(t)Sx(t) - x^T(t-\tau)Sx(t-\tau)]dt \\
= \; & V(0, x(0)) - V(t_f, x(t_f)) + \int_0^{t_f} [x^T(t)Qx(t)
\end{aligned}$$

$$
\begin{aligned}
& +x^T(t)(A-B_uK)^TPx(t)+x^T(t-\tau)A_d^TPx(t)+x^T(t)\Delta A^TPx(t) \\
& +x^T(t-\tau)\Delta A_d^TPx(t)-x^T(t)K^T\Delta B_u^TPx(t)+w^T(t)B_w^TPx(t) \\
& +x^T(t)P(A-B_uK)x(t)+x^T(t)PA_dx(t-\tau) \\
& +x^T(t)P\Delta Ax(t)+x^T(t)P\Delta A_dx(t-\tau)-x^T(t)P\Delta B_uKx(t) \\
& +x^T(t)PB_ww(t)+x^T(t)Sx(t)-x^T(t-\tau)Sx(t-\tau)]dt \\
= \; & V(0,x(0))-V(t_f,x(t_f))+\int_0^{t_f}\{x^T(t)[Q+S+(A-BK)^TP+P(A-BK)]x(t) \\
& +x^T(t-\tau)A_d^TPx(t)+x^T(t)PA_dx(t-\tau)-x^T(t-\tau)Sx(t-\tau) \\
& +(x^T(t)\Delta A^TPx(t)+x^T(t)P\Delta Ax(t))+(x^T(t-\tau)\Delta A_d^TPx(t)+x^T(t)PA_dx(t-\tau)) \\
& -(x^T(t)K^T\Delta B_u^TPx(t)+x^T(t)P\Delta B_uKx(t)) \\
& +x^T(t)PB_ww(t)+w^T(t)B_w^TPx(t)-\rho^2w^T(t)w(t)-\frac{1}{\rho^2}x^T(t)PB_wB_w^TPx(t) \\
& +\rho^2w^T(t)w(t)+\frac{1}{\rho^2}x^T(t)PB_wB_w^TPx(t)\}dt \\
\leq \; & V(0,x(0))+\int_0^{t_f}\{x^T(t)[Q+S+(A-B_uK)^TP+P(A-B_uK)+\frac{1}{\rho^2}PB_wB_w^TP]x(t) \\
& +x^T(t-\tau)A_d^TPx(t)+x^T(t)PA_dx(t-\tau)-x^T(t-\tau)Sx(t-\tau) \\
& +(x^T(t)A_p^TA_px(t)+x^T(t)PPx(t))+(x^T(t-\tau)A_{dp}^TA_{dp}x(t-\tau)+x^T(t)PPx(t)) \\
& +(x^T(t)PPx(t)+x^T(t)K^TB_p^TB_pKx(t))-\left(\rho w(t)-\frac{1}{\rho}B_w^TPx(t)\right)^T\left(\rho w(t)-\frac{1}{\rho}B_w^TPx(t)\right) \\
& +\rho^2w^T(t)w(t)\}dt \\
\leq \; & V(0,x(0))+\int_0^{t_f}\{x^T(t)[Q+S+(A-B_uK)^TP+P(A-B_uK)+\frac{1}{\rho^2}PB_wB_w^TP \\
& +3PP+A_p^TA_p+K^TB_p^TB_pK]x(t) \\
& +x^T(t-\tau)A_d^TPx(t)+x^T(t)PA_dx(t-\tau)+x^T(t-\tau)A_{dp}^TA_{dp}x(t-\tau) \\
& -x^T(t-\tau)Sx(t-\tau)+\rho^2w^T(t)w(t)\}dt
\end{aligned}
\tag{13}
$$

Then, we get the following result.

Theorem : *In the time-delay system (1), suppose there exists a positive definite matrix $P = P^T > 0$ is the solution of following linear matrix inequalities(LMIs) are satisfied*

$$
\begin{bmatrix}
\begin{pmatrix} A^TP+PA-K^TB_u^TP-PB_uK+S+Q \\ +\frac{1}{\rho^2}PB_wB_w^TP+3PP+A_p^TA_p+K^TB_p^TB_pK \end{pmatrix} & PA_d \\
A_d^TP & -S+A_{dp}^TA_{dp}
\end{bmatrix} < 0
\tag{14}
$$

Then the H_∞ control performance of (10) is guaranteed for a prescribed ρ. ■

For the proof of Theorem, the following lemma is necessary.

Lemma [21]: *For any matrices (or vectors) X and Y with appropriate dimensions, we have*

$$
X^TY+Y^TX \leq X^TEX+Y^TE^{-1}Y
\tag{15}
$$

where E is any positive-definite symmetric matrix. ■

Proof of Theorem: By (13) and (15), we obtain

$$
\begin{aligned}
& \int_0^{t_f} x^T(t)Qx(t)dt \\
\leq\ & V(0,x(0)) + \int_0^{t_f} \{x^T(t)[Q+S+(A-B_uK)^TP+P(A-B_uK)+\frac{1}{\rho^2}PB_wB_w^TP \\
& +3PP+A_p^TA_p+K^TB_p^TB_pK]x(t) \\
& +x^T(t-\tau)A_d^TPx(t)+x^T(t)PA_dx(t-\tau)+x^T(t-\tau)A_{dp}^TA_{dp}x(t-\tau) \\
& -x^T(t-\tau)Sx(t-\tau)+\rho^2w^T(t)w(t)\}dt \\
\leq\ & V(0,x(0)) + \int_0^{t_f} \{ \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \\
& \begin{bmatrix} \begin{pmatrix} A^TP+PA-K^TB_u^TP-PB_uK+S+Q \\ +\frac{1}{\rho^2}PB_wB_w^TP+3PP+A_p^TA_p+K^TB_p^TB_pK \end{pmatrix} & PA_d \\ A_d^TP & -S+A_{dp}^TA_{dp} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \\
& +\rho^2w^T(t)w(t)\}dt
\end{aligned}
$$

From (14), we get

$$
\begin{aligned}
& \int_0^{t_f} x^T(t)Qx(t)dt \\
\leq\ & V(0,x(0)) + \rho^2 \int_0^{t_f} w^T(t)w(t)dt \\
=\ & x^T(0)Px(0) + \int_{0-\tau}^{0} x^T(\theta)Sx(\theta)d\theta + \rho^2 \int_0^{t_f} w^T(t)w(t)dt \\
=\ & x^T(0)Px(0) + \tau x^T(0)Sx(0) + \rho^2 \int_0^{t_f} w^T(t)w(t)dt
\end{aligned}
$$

Therefore, the H_∞ control performance is achieved with a prescribed ρ. ■

By introducing a new matrix

$$
Z = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{16}
$$

where $W = P^{-1}$ and multiplying it into (14), we get

$$
\begin{bmatrix} \begin{pmatrix} WA^T+AW-WK^TB_u^T-B_uKW+\frac{1}{\rho^2}B_wB_w^T \\ +W(S+Q+A_p^TA_p)W+3I+WK^TB_p^TB_pKW \end{pmatrix} & A_d \\ A_d^T & -S+A_{dp}^TA_{dp} \end{bmatrix} < 0 \tag{17}
$$

With $Y = KW$ and by the Schur complements [19], the matrix inequality (17) can be rearranged as the following forms:

$$\begin{bmatrix} \begin{pmatrix} AW+WA^T-B_uY-(B_uY)^T \\ +\frac{1}{\rho^2}B_wB_w^T+3I \end{pmatrix} & W & B_pY & A_d \\ W & -(S+Q+A_p^TA_p)^{-1} & 0 & 0 \\ (B_pY)^T & 0 & -I & 0 \\ A_d^T & 0 & 0 & -S+A_{dp}^TA_{dp} \end{bmatrix} < 0 \tag{18}$$

By the change of variable $\nu=-\frac{1}{\rho^2}$, (18) is equivalent the following LMI:

$$\begin{bmatrix} \begin{pmatrix} AW+WA^T-B_uY \\ -(B_uY)^T-\nu B_wB_w^T+3I \end{pmatrix} & W & B_pY & A_d \\ W & -(S+Q+A_p^TA_p)^{-1} & 0 & 0 \\ (B_pY)^T & 0 & -I & 0 \\ A_d^T & 0 & 0 & -S+A_{dp}^TA_{dp} \end{bmatrix} < 0 \tag{19}$$

The parameters W and Y (thus $P^{-1}=W$ and $K=YW^{-1}$) can be obtained by solving the LMIP in (18) for a prescribed attenuation level ρ^2.

To obtain better robust H_∞ performance in (10), we can minimize ν subject to (19) as the following EVP:

$$\min_{\{W,Y\}} \quad \nu$$
$$\text{subject to} \quad P=P^T>0,\ \nu<0 \text{ and (19).}$$

B. H_2 Control Design

An LQ performance related to the state and control action is considered as

$$J_2(0,t_f)=\int_0^{t_f}(x^T(t)Q_2x(t)+u^T(t)R_2u(t))dt \tag{20}$$

where $Q_2>0$ and $R_2>0$. A straightforward objective is to minimize this performance. However, this objective is not easily tractable, since uncertainties involve in state dynamics. Therefore, in this study, a suboptimal approach is taken by minimizing the upper bound of performance index. In this case of $w(t)=0$, we obtain

$$\begin{aligned} & J_2(0,t_f) \\ = \; & V(0,x(0))-V(t_f,x(t_f))+\int_0^{t_f}[x^T(t)Q_2x(t)+u^T(t)R_2u(t)+\dot V(t,x(t))]dt \\ = \; & V(0,x(0))-V(t_f,x(t_f))+\int_0^{t_f}\{x^T(t)Q_2e(t)+x^T(t)K^TR_2Kx(t)+x^T(t)(A-B_uK)^TPx(t) \\ & +x^T(t)P(A-B_uK)x(t)+x^T(t)\Delta A^TPx(t)+x^T(t)P\Delta Ax(t)+x^T(t-\tau)A_d^TPx(t) \\ & +x^T(t)PA_dx(t-\tau)+x^T(t-\tau)\Delta A_d^TPx(t)+x^T(t)P\Delta A_dx(t-\tau)-x^T(t)K^T\Delta B_u^TPx(t) \\ & -x^T(t)P\Delta B_uKx(t)+x^T(t)Sx(t)-x^T(t-\tau)Sx(t-\tau)\}dt \end{aligned}$$

$$\begin{aligned}
&\leq x^T(0)Px(0)+\tau x^T(0)Sx(0)+\int_0^{t_f}\{x^T(t)[Q_2+K^TR_2K+P(A-B_uK)+(A-B_uK)^TP]x(t)\\
&\quad +x^T(t)PA_dx(t-\tau)+x^T(t-\tau)A_d^TPx(t)+x^T(t)\Delta A^T\Delta Ax(t)+x^T(t)PPx(t)\\
&\quad +x^T(t-\tau)\Delta A_d^T\Delta A_dx(t-\tau)+x^T(t)PPx(t)\\
&\quad +x^T(t)K^T\Delta B_u^T\Delta B_uKx(t)+x^T(t)PPx(t)+x^T(t)Sx(t)-x^T(t-\tau)Sx(t-\tau)\}dt\\
&\leq x^T(0)Px(0)+\tau x^T(0)Sx(0)+\int_0^{t_f}\{x^T(t)[Q_2+K^TR_2K+P(A-B_uK)+(A-B_uK)^TP\\
&\quad +S+3PP]x(t)+x^T(t)PA_dx(t-\tau)+x^T(t-\tau)A_d^TPx(t)+x^T(t)A_p^TA_px(t)\\
&\quad +x^T(t-\tau)A_{dp}^TA_{dp}x(t-\tau)+x^T(t)K^TB_p^TB_pKx(t)-x^T(t-\tau)Sx(t-\tau)\}dt\\
&= x^T(0)Px(0)+\tau x^T(0)Sx(0)+\int_0^{t_f}\{\begin{bmatrix} x(t)\\ x(t-\tau)\end{bmatrix}^T\cdot\\
&\quad \begin{bmatrix}\begin{pmatrix} Q_2+K^TR_2K+P(A-B_uK)+A_p^TA_p\\ +K^TB_p^TB_pK+(A-B_uK)^TP+S+3PP\end{pmatrix} & PA_d\\ A_d^TP & -S+A_{dp}^TA_{dp}\end{bmatrix}\begin{bmatrix} x(t)\\ x(t-\tau)\end{bmatrix}\}dt
\end{aligned}$$

Note that if

$$\begin{bmatrix}\begin{pmatrix} Q_2+K^TR_2K+P(A-B_uK)+A_p^TA_p\\ +K^TB_p^TB_pK+(A-B_uK)^TP+S+3PP\end{pmatrix} & PA_d\\ A_d^TP & -S+A_{dp}^TA_{dp}\end{bmatrix}<0, \tag{21}$$

then we obtain

$$J_2(0,t_f)\leq x^T(0)Px(0)+\tau x^T(0)Sx(0) \tag{22}$$

where $x(0)$ are initial value of $x(t)$.

Since the latter term $\tau x^T(0)Sx(0)$ in (22) is some constant and according to analysis above, we can formulate the suboptimal control problem with H_2 performance by minimizing the upper bound of $J_2(0,t_f)$ subject to (21) as follows:

$$\begin{aligned}
&\min_P && x^T(0)Px(0)\\
&\text{subject to} && P=P^T>0 \text{ and } (21)
\end{aligned} \tag{23}$$

By introducing a new matrix

$$Z=\begin{bmatrix} W & 0\\ 0 & I\end{bmatrix}=\begin{bmatrix} P^{-1} & 0\\ 0 & I\end{bmatrix} \tag{24}$$

where $W=P^{-1}$ and multiplying it into (21), we get

$$\begin{bmatrix}\begin{pmatrix} WA^T+AW-WK^TB_u^T-B_uKW+WK^TR_2KW\\ +W(S+Q_2+A_p^TA_p)W+3I+WK^TB_p^TB_pKW\end{pmatrix} & A_d\\ A_d^T & -S+A_{dp}^TA_{dp}\end{bmatrix}<0 \tag{25}$$

With $Y = KW$ and by the Schur complements [19], the matrix inequality (17) can be rearranged as the following forms:

$$\begin{bmatrix} \begin{pmatrix} WA^T + AW - B_uY \\ -Y^TB_u^T + 3I \end{pmatrix} & W & B_pY & Y & A_d \\ W & -(S+Q_2+A_p^TA_p)^{-1} & 0 & 0 & 0 \\ (B_pY)^T & 0 & -I & 0 & 0 \\ Y^T & 0 & 0 & -R_2^{-1} & 0 \\ A_d^T & 0 & 0 & 0 & -S+A_{dp}^TA_{dp} \end{bmatrix} < 0 \tag{26}$$

Therefore, the suboptimal H_2 problem in (23) can be reformulated as

$$\begin{aligned} &\min_{\{W,Y\}} && e^T(0)W^{-1}e(0) \\ &\text{subject to} && W = W^T > 0 \text{ and } (26). \end{aligned} \tag{27}$$

Note that the minimization problem in (27) is not the standard form of the LMI problem. However, the minimization problem in (27) can be transformed into an LMI problem as the following procedure. By introducing a new variable σ such that

$$e^T(0)W^{-1}e(0) < \sigma \tag{28}$$

Note that (28) is equivalent to the following LMI:

$$\begin{bmatrix} \sigma & e^T(0) \\ e(0) & W \end{bmatrix} > 0 \tag{29}$$

Therefore, $J_2(0, t_f) < \sigma + \tau x^T(0)Sx(0)$ and the minimization problem in (27) can be transformed into the following LMI problem:

$$\begin{aligned} &\min_{\{W,Y\}} && \sigma \\ &\text{subject to} && W = W^T > 0,\ \sigma > 0,\ (23) \text{ and } (29). \end{aligned} \tag{30}$$

C. Mixed H_2/H_∞ Control Design

The mixed H_2/H_∞ Control Design can be characterized as the following EVP:

$$\begin{aligned} &\min_{\{W,Y\}} && e^T(0)W^{-1}e(0) \\ &\text{subject to} && W = W^T > 0,\ (\mathbf{??}) \text{ and } (30) \end{aligned} \tag{31}$$

From the analysis above, the most important task in this study is to find the W and Y by solving the EVP in (31) for mixed H_2/H_∞ control.

The mixed H_2/H_∞ control design is summarized as follows.

Design Procedure:

Step 1: Select weighting matrices Q, Q_2, R_2 and S according to the design purpose.

Step 2: Select the bounding matrices A_p, A_{dp} and B_p.

Step 3: Solve the EVP for mixed H_2/H_∞ controller in (31) to obtain W and Y (thus $P = W^{-1}$ and $K = YW^{-1}$ can also be obtained).

Step 4: Check the assumptions of

$$\begin{aligned} \|\Delta A x(t)\| &\leq \|A_p x(t)\|, \\ \|\Delta A_d x(t-\tau)\| &\leq \|A_{dp} x(t-\tau)\|, \end{aligned}$$

and

$$\|\Delta B_u K x(t)\| \leq \|B_p K x(t)\|.$$

If they are not satisfied, adjust (expand) the bounds for all elements in A_p, A_{dp} and B_p, and then repeated Steps 3-4.

Step 5: Obtain the mixed H_2/H_∞ controller in (2).

IV. Simulation Example

A. Example 1

To illustrate the proposed mixed H_2/H_∞ control approach, we consider the second-order example as follows [8]:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + B_u u(t) + \Delta A x(t) + \Delta A_d x(t-\tau) \\ &\quad + \Delta B_u u(t) + B_w w(t) \end{aligned} \tag{32}$$

where

$$\begin{aligned} x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \; A = \begin{bmatrix} -2.0 & 0 \\ 0.6 & 1 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.8 & 0 \\ -0.6 & -2.4 \end{bmatrix}, \\ B_u &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_w = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \\ \Delta A &= \begin{bmatrix} 0.5x_1(t)\sin(x_1(t)) \\ 0.2x_1(t)x_2(t) \end{bmatrix}, \; \Delta B_u = \begin{bmatrix} 0 \\ 0.3x_2(t)\sin(x_1(t)) \end{bmatrix}, \\ \Delta A_d &= \begin{bmatrix} -0.1x_1(t)\sin(x_1(t)) \\ 0.2x_2(t)\sin(x_2(t)) \end{bmatrix}, \end{aligned}$$

and

$$\tau = 2.4731.$$

Now, following the Design Procedure in the above section, the mixed H_2/H_∞ control design is given by the following steps.

Step 1: Select weighting matrices Q, Q_2, R_2 and S as follows:

$$Q = Q_2 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix},\ R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Step 2: The bounding matrices are chosen as

$$\begin{aligned} A_p &= 0.3 \times \begin{bmatrix} -2.0 & 0 \\ 0.6 & 1 \end{bmatrix}, \\ B_p &= 0.3 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned}$$

and

$$A_{dp} = 0.2 \times \begin{bmatrix} -0.8 & 0 \\ -0.6 & -2.4 \end{bmatrix}.$$

The assumptions of

$$\begin{aligned} \|\Delta A x(t)\| &\leq \|A_p x(t)\|, \\ \|\Delta A_d x(t-\tau)\| &\leq \|A_{dp} x(t-\tau)\|, \end{aligned}$$

and

$$\|\Delta B_u K x(t)\| \leq \|B_p K x(t)\|$$

are satisfied (refer to Fig. 4).

Step 3: Solve the EVP using the LMI optimization toolbox in Matlab [20]. In this case, $\rho_{\min} = 0.1417$,

$$W = \begin{bmatrix} 1.7996 & -0.2928 \\ -0.2928 & 0.0950 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 1.1142 & 3.4327 \\ 3.4327 & 21.0969 \end{bmatrix}.$$

Step 4: The control parameter is found to be

$$K = \begin{bmatrix} 39.2549 & 237.2479 \end{bmatrix}.$$

Figs. 1 to 4 present the simulation results from initial conditions assumed to be $(x_1(0), x_2(0))^T = (2, -1)^T$.

Fig. 1 shows the trajectories of the states $x_1(t)$ and $x_2(t)$. The control signal $u(t)$ is presented in Fig. 2. Fig. 3 shows the external disturbance $w(t)$. The simulation example results have shown that the mixed H_2/H_∞ control design can be easily accomplished by the proposed design procedure and the robust performance of time delay systems can be efficiently achieved by the proposed mixed H_2/H_∞ control scheme.

In this example, we consider the system with uncertain parameters. With the same amount of time delay, $\tau = 2.4731$, our method can guarantee the system's robustness. Notice that the method proposed in [8] do not allow any uncertain parameters.

B. Example 2

To illustrate the proposed mixed H_2/H_∞ control approach, we consider the second-order example as follows [9]:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_d x(t-\tau) + B_u u(t) + \Delta A x(t) + \Delta A_d x(t-\tau) \\ &\quad + \Delta B_u u(t) + B_w w(t)\end{aligned} \tag{33}$$

where

$$\begin{aligned}x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ A = \begin{bmatrix} -2.0 & 0 \\ 0 & -2.5 \end{bmatrix},\ A_d = \begin{bmatrix} -1.0 & 0 \\ -1.0 & -1.0 \end{bmatrix}, \\ B_u &= \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \Delta A &= \begin{bmatrix} 0.5\cos(t) & 0 \\ 0 & 0.6\sin(t) \end{bmatrix},\ \Delta B_u = \begin{bmatrix} 0 \\ 0.3x_1(t)\sin(t) \end{bmatrix}, \\ \Delta A_d &= \begin{bmatrix} 0.5\cos(t) & 0 \\ 0 & 0.6\sin(t) \end{bmatrix},\end{aligned}$$

and

$$\tau = 15\,\mathrm{sec}\,.$$

Now, following the Design Procedure in the above section, the mixed H_2/H_∞ control design is given by the following steps.

Step 1: Select weighting matrices Q, Q_2, R_2 and S as follows:

$$Q = Q_2 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix},\ R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Step 2: The bounding matrices are chosen as

$$A_p = 0.28 \times \begin{bmatrix} -2.0 & 0 \\ 0 & -2.5 \end{bmatrix},$$

$$B_p = 0.45 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$A_{dp} = 0.5 \times \begin{bmatrix} -1.0 & 0 \\ -1.0 & -1.0 \end{bmatrix}.$$

The assumptions of

$$\begin{aligned} \|\Delta A x(t)\| &\le \|A_p x(t)\|, \\ \|\Delta A_d x(t-\tau)\| &\le \|A_{dp} x(t-\tau)\|, \end{aligned}$$

and

$$\|\Delta B_u K x(t)\| \le \|B_p K x(t)\|$$

are satisfied (refer to Fig. 8).

Step 3: Solve the EVP using the LMI optimization toolbox in Matlab [20]. In this case, $\rho_{\min} = 0.020004$,

$$W = \begin{bmatrix} 1.5616 & -0.1128 \\ -0.1128 & 32.0734 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 0.6405 & 0.0023 \\ 0.0023 & 0.0312 \end{bmatrix}.$$

Step 4: The control parameter is found to be

$$K = \begin{bmatrix} 3.2465 & 43.3555 \end{bmatrix}.$$

Figs. 5 to 8 present the simulation results from initial conditions assumed to be $(x_1(0),\, x_2(0))^T = (2,\, -1)^T$.

Fig. 5 shows the trajectories of the states $x_1(t)$ and $x_2(t)$. The control signal $u(t)$ is presented in Fig. 6. Fig. 7 shows the external disturbance $w(t)$. The simulation example results have shown that the mixed H_2/H_∞ control design can be easily accomplished by the proposed design procedure and the robust performance of time delayed systems can be efficiently achieved by the proposed mixed H_2/H_∞ control scheme.

Applying our method to this uncertain time delay system, it is found that this system remain robust stability even when the time delay is equal to 15 sec. We note that the result is much better than [9] for time delay between $0 \le \tau(t) \le 11.0136$ sec.

V. Conclusions

In this paper, the mixed H_2/H_∞ control technique has been provided to achieve robust performance for uncertain time delay systems. Actually, the proposed robust control can be applied to any robust control design of uncertain time delay system. By employing the H_∞ attenuation technique, the performance of robust control design for uncertain time delay systems can be significantly improved. Furthermore, the robust control scheme is also developed to eliminate as much as possible the effect of the external disturbance. Therefore, the proposed design algorithm is appropriate for practical control design with bounded external disturbances. The outcome of the mixed H_2/H_∞ performance problems in this study is characterized in terms of two eigenvalue problems (EVPs). The EVPs can be solved very efficiently by the convex optimization techniques.

The proposed design procedures are very simple and can be performed efficiently by the Matlab toolbox. Simulation results indicate that the desired mixed H_2/H_∞ performance for uncertain time delay systems can be achieved via the proposed method.

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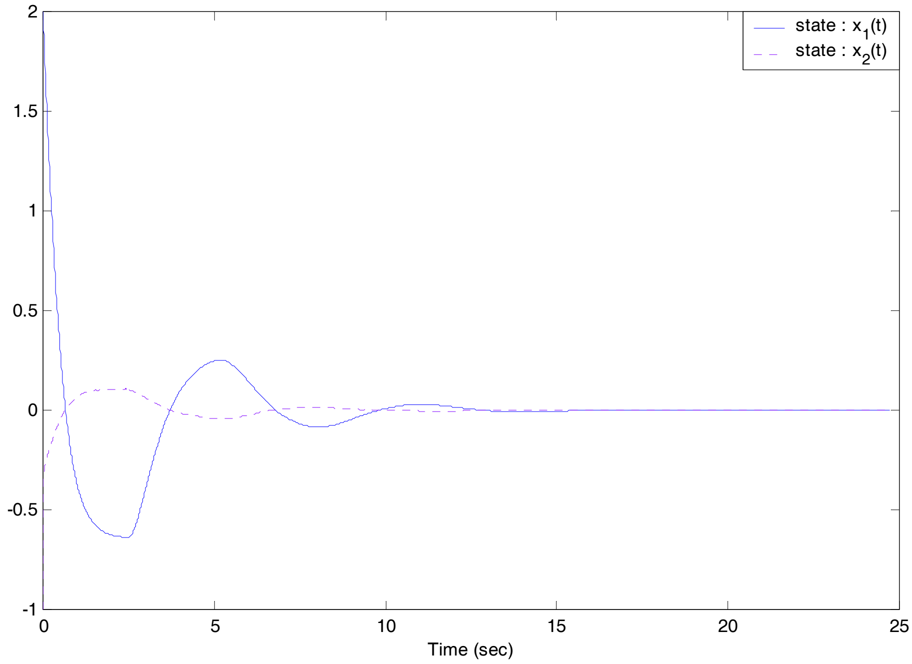

Fig. 1 The trajectories of the states $x_1(t)$ and $x_2(t)$.

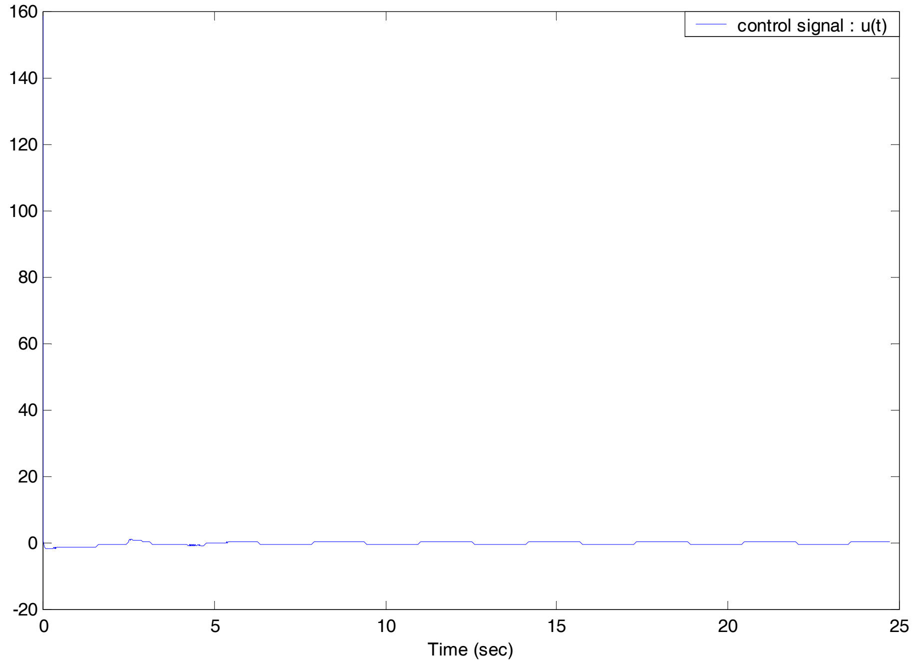

Fig. 2 The control input $u(t)$.

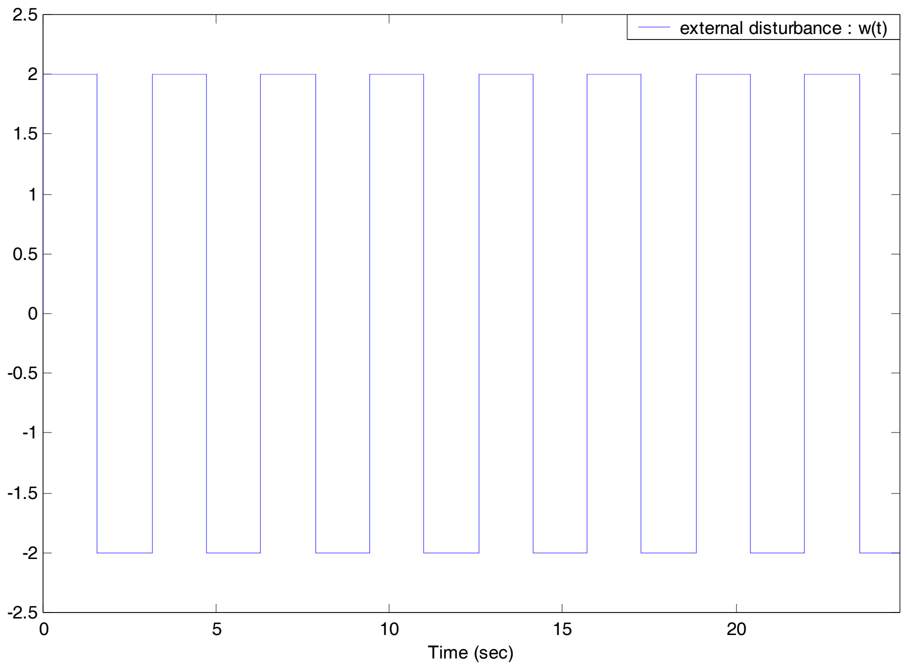

Fig. 3 The external disturbance $w(t)$.

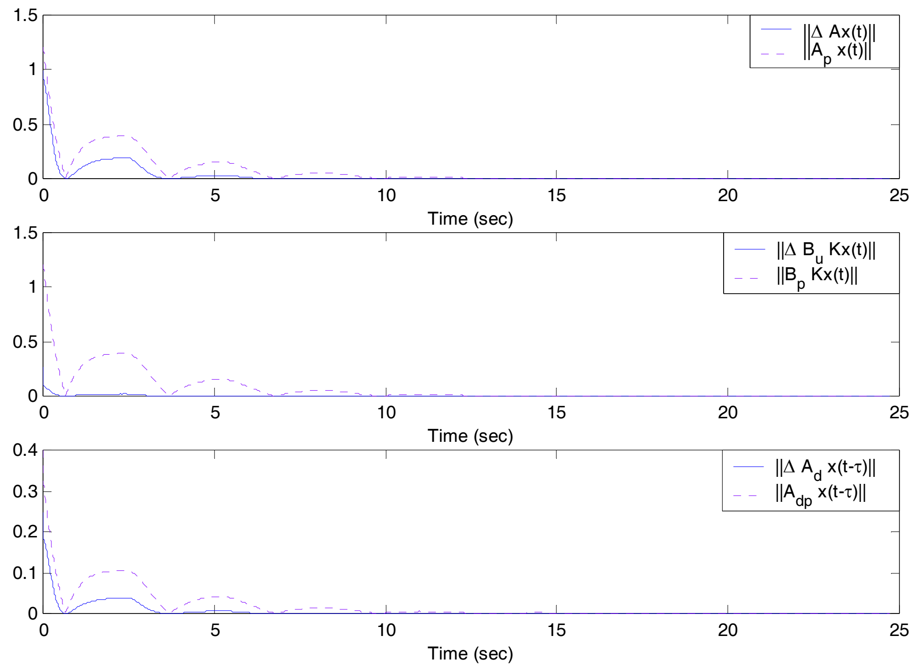

Fig. 4 The plots $\|\Delta Ax(t)\|$ and $\|A_p x(t)\|$, $\|\Delta B_u Kx(t)\|$ and $\|B_p Kx(t)\|$,

$\|\Delta A_d x(t-\tau)\|$ and $\|A_{dp} x(t-\tau)\|$.

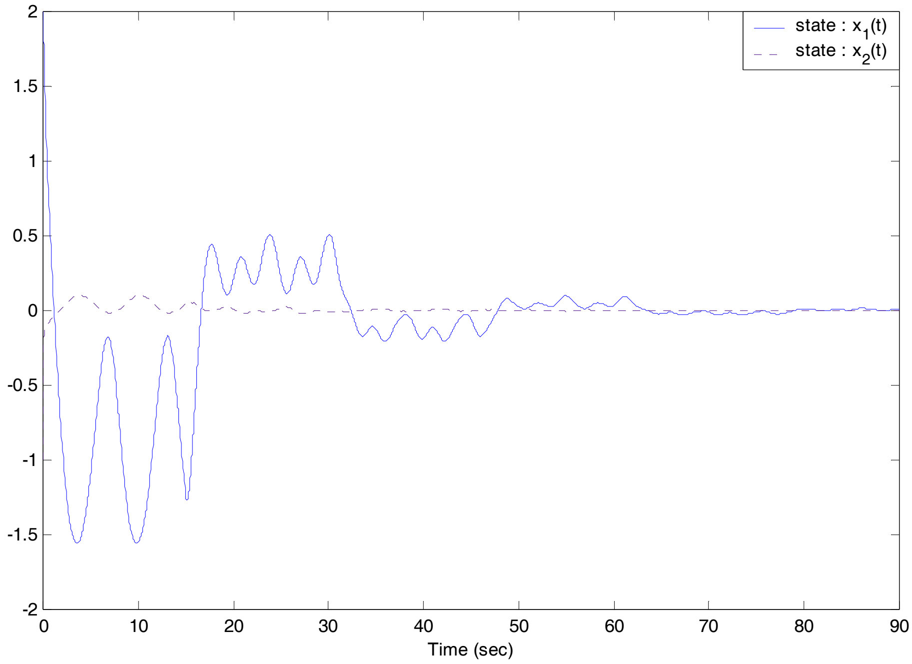

Fig. 5 The trajectories of the states $x_1(t)$ and $x_2(t)$.

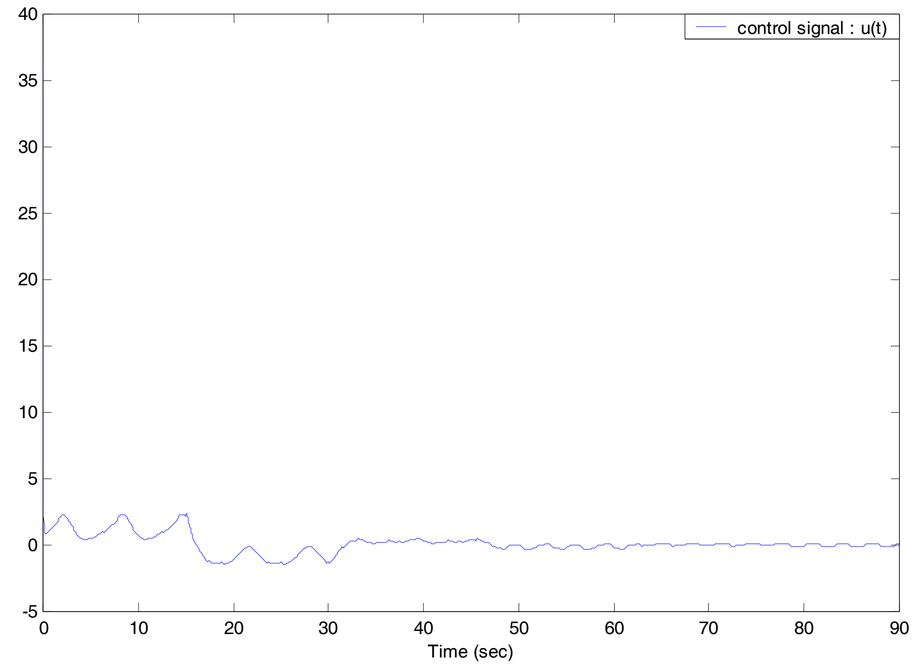

Fig. 6 The control input $u(t)$.

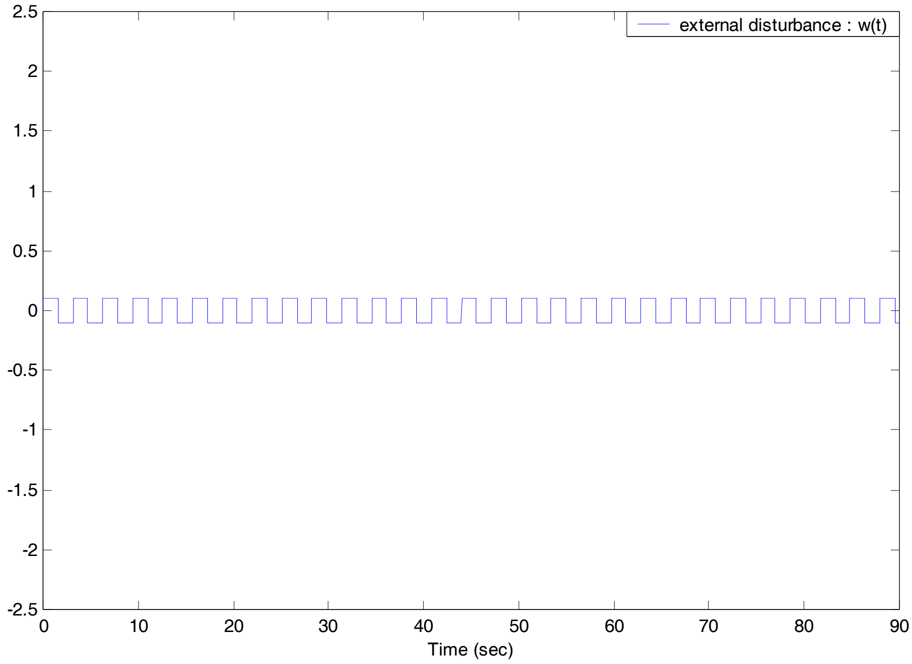

Fig. 7 The external disturbance $w(t)$.

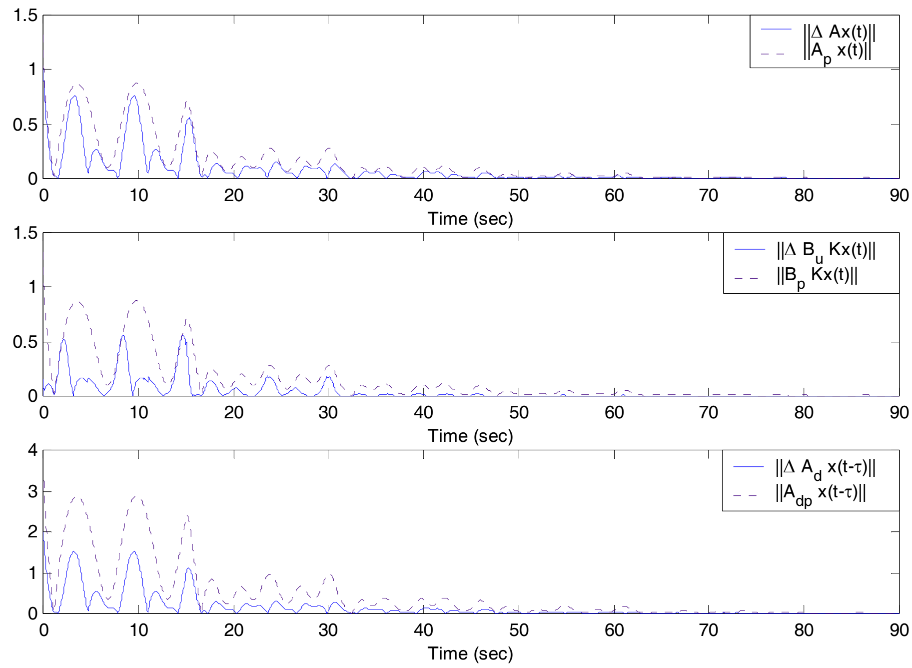

Fig. 8 The plots $\|\Delta Ax(t)\|$ and $\|A_p x(t)\|$, $\|\Delta B_u Kx(t)\|$ and $\|B_p Kx(t)\|$,

$\|\Delta A_d x(t-\tau)\|$ and $\|A_{dp} x(t-\tau)\|$.